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Topology and its Applications 154 (2007) 2040–2049

Topology and its Applications

www.elsevier.com/locate/topol

Representable topologies and locally connected spaces

M.J. Campión [a], J.C. Candeal [b], E. Induráin [a,*], G.B. Mehta [c]

[a] *Universidad Pública de Navarra, Departamento de Matemátics, Campus Arrosadía, Edificio "Las Encinas", E-31006 Pamplona, Spain*

[b] *Universidad de Zaragoza, Facultad de Ciencias Económicas y Empresariales, Departamento de Análisis Económico, c/ Doctor Cerrada 1-3, E-50005 Zaragoza, Spain*

[c] *University of Queensland, Department of Economics, 4072 Brisbane, Queensland, Australia*

Received 1 September 2005; accepted 1 May 2006

Abstract

The objective of this paper is to study certain order-theoretic properties of locally connected topologies. In the main theorem we prove that the assertion that a locally connected topology which satisfies the countable chain condition has the continuous representability property is undecidable in ZFC set theory. Some related problems and generalizations are also included.

MSC: primary 54F05; secondary 06A06, 91B16

Keywords: Totally preordered structures; Continuous numerical representations; Locally connected separable spaces; Representable topologies

1. Introduction

Let X be a set. Then a topology τ on X is said to have the *continuous representability property* (CRP) if for every τ-continuous total preorder $\precsim$ on X there is an order-preserving real-valued function f on X, that is, a real-valued function f with the property that $x \precsim y$ if and only if $f(x) \leqslant f(y)$. The problem of proving the existence of such an order-preserving function is called a (continuous) *representation problem*. There is now a vast literature which deals with representation problems.

The object of this paper is to study some well-known topological properties from the representation point of view. In particular, this paper deals with topologies on a set X that are locally connected and other topologies which are closely related to locally connected topologies. We also consider generalizations of locally connected spaces. In general, a locally connected topological space need not have the CRP. Therefore, we look for additional conditions which ensure continuous representability or non-representability. In this connection we study the countable chain condition and the topological separability condition. It is important to try to find conditions which are simple and are widely used in the field of topology and its applications. The conditions of local connectedness and weak local connectedness

* Corresponding author. Tel.: +34 948 169551; fax: +34 948 166057.

E-mail addresses: mjesus.campion@unavarra.es (M.J. Campión), candeal@unizar.es (J.C. Candeal), steiner@unavarra.es (E. Induráin), g.mehta@economics.uq.edu.au (G.B. Mehta).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.05.011

that we study in this paper are two such conditions which satisfy both these criteria. The same can also be said about the countable chain condition and the separability conditions.

We start by considering a topological space X with a topology τ that is locally connected and satisfies the countable chain condition. Our first result is rather surprising. We show that it is not possible to prove in the framework of the ZFC axioms whether or not such a topology τ has the continuous representation property. We do this by establishing a connection with the famous Souslin Hypothesis of set theory. It is well known that the Souslin Hypothesis is undecidable in ZFC set theory (see, e.g. Roitman [21, Chapter 7], or Ciesielski [8, Section 8.3]). We then show that if we strengthen the countable chain condition we do get a positive result. We prove that a locally connected and separable topology has the CRP.

The proof given here uses Eilenberg's representation theorem and is based on the concept of inductive limits. For a different proof of this theorem which does not use Eilenberg's theorem we refer the reader to Candeal et al. [7]. We then give several generalizations of this theorem. We also study the problem of finding semicontinuous representations. In the last section, we study generalizations of locally connected topologies. We prove a continuous representation theorem for topological spaces that are *locally connected im kleinen* or *weakly locally connected*.

2. Preliminaries

A *preorder* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive. If $\precsim$ is a preorder on a set X, then we will refer to the pair $(X, \precsim)$ as a preordered set.

An anti-symmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $[x \precsim y]$ or $[y \precsim x]$. A totally ordered set $(X, \precsim)$ is also said to be a *chain*.

If $\precsim$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $[x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)]$ and $[x \sim y \iff (x \precsim y) \wedge (y \precsim x)]$. Also, the associated *dual* preorder $\precsim_d$ is defined by $[x \precsim_d y \iff y \precsim x]$.

Let $(X, \precsim)$ be a totally preordered set and let $X/\sim$ be the set of equivalence classes. If $x \in X$ we denote the equivalence class of x by $[x]$. The preorder $\precsim$ on X induces a natural order $\preceq$ on $X/\sim$ defined by $[x] \preceq [y] \iff x \precsim y$. Let $[x], [y]$ be two equivalence classes in $X/\sim$. Then we say that the ordered pair $([x], [y]) \in (X/\sim) \times (X/\sim)$ is a *jump* if there is no $[z] \in X/\sim$ such that $[x] < [z] < [y]$, where $<$ denotes the asymmetric part of $\preceq$. If $([x], [y])$ is a jump then we sometimes abuse notation and say that (x, y) is a jump in X. But it should be remembered that a jump is only defined for the corresponding equivalence classes.

A totally ordered set $(X, \precsim)$ is said to be *dense* if it has no jumps. A subset Z of X is said to be *order-dense* in X if $x, y \in X$ and $x \prec y$ imply that there exists $z \in Z$ such that $x \precsim z \precsim y$. $(X, \precsim)$ is said to be *order-separable* if it has a countable order-dense subset.

A totally ordered set $(X, \precsim)$ is said to be *Dedekind-complete* if each nonempty subset F that has an upper bound has a least upper bound. It is known that each totally ordered set has an essentially unique Dedekind completion. (See Gillman and Jerison [15, p. 3].) A totally ordered set is said to be a *continuum* if it is dense and Dedekind-complete.

A subset A of a totally preordered set $(X, \precsim)$ is said to be *decreasing* if for each $x \in A$ and $y \in X$ such that $y \precsim x$ it follows that $y \in A$.

Let $(X, \precsim)$ be a totally preordered set. The family of all sets of the form $L(x) = \{a \in X\colon\ a \prec x\}$ and $G(x) = \{a \in X\colon\ x \prec a\}$, where $x \in X$ is a sub-base for a topology $\tau_{\precsim}$ on X. The pair $(X, \tau_{\precsim})$ is called the *order topology* on X. The pair $(X, \tau_{\precsim})$ is called a *preordered topological space*.

If $(X, \precsim)$ is a preordered set and τ is a topology on X, then the preorder $\precsim$ is said to be τ-upper semicontinuous on X (respectively, τ-lower semicontinuous on X) if for each $x \in X$ the set $\{a \in X\colon\ x \precsim a\}$ (respectively, $\{a \in X\colon\ a \precsim x\}$) is τ-closed in X. It is said to be τ*-continuous* on X if it is both τ-upper semicontinuous and τ-lower semicontinuous. A topology on a totally preordered set $(X, \precsim)$ is said to be a *natural* topology if it is finer than the order topology, that is, if the preorder $\precsim$ is τ-continuous.

If $(X, \precsim)$ is a preordered set then a real-valued function f on X is said to be

(i) *increasing* if for every $x, y \in X$, $[x \precsim y \Longrightarrow f(x) \leqslant f(y)]$;
(ii) *order-preserving* if f is increasing and $[x \prec y \Longrightarrow f(x) < f(y)]$.

An order-preserving function is also said to be an *order-monomorphism* or an *order-embedding*. We note that even if an order-preserving function on $(X, \precsim)$ is not injective, then it induces an injective order preserving function on the quotient set of equivalence classes. A bijective order-monomorphism is an *order-isomorphism*. If a set X is endowed with a topology τ then the total preorder $\precsim$ on X is said to be *continuously representable* if there exists an order-embedding that is continuous with respect to the topology τ and the usual topology on the real line $\mathbb{R}$.

A totally preordered set $(X, \precsim)$ is said to be *scattered* if there is no order-embedding of the rationals $\mathbb{Q}$ into X. It is said to be *short* if there is no order-embedding of the first uncountable ordinal ω_1 or its dual into X.

A topological space (X, τ) is said to be *separable* if there exists a countable subset $D \subseteq X$ that meets every nonempty τ-open subset of X. A topological space (X, τ) is said to be *connected* if there is no partition of X into two disjoint nonempty τ-open subsets.

A totally ordered set $(X, \precsim)$ is said to satisfy the *countable chain condition* (ccc) if each family of pairwise disjoint open order-intervals is countable. More generally a topological space (X, τ) is said to satisfy ccc if each family of pairwise disjoint τ-open sets is countable. In these cases, we also say that the ordered set or the topological space has the *Souslin property*.

A topological space (X, τ) is said to be *locally connected at a point* x if x has a basis of connected open neighbourhoods, and it is said to be *locally connected* if it is locally connected at every $x \in X$. This is equivalent to the assertion that every connected component (i.e.: maximal connected subset) of an open subset $O \subseteq X$ is open. (See Dugundji [12, p. 113].)

A topological space (X, τ) such that all the connected components of the whole set X are open is called a *weakly locally connected space*.

The topology τ on X is said to have the *continuous representation property* (CRP) if every continuous total preorder $\precsim$ defined on X admits a continuous representation. For notational economy we will call such a topology a *“representable topology”*.

A topological space (X, τ) is said to be *locally connected im kleinen* (i.e., “at a small scale”, in German) *at the point* x if for each open neighbourhood U of x there is a neighbourhood V of x such that if $y \in V$ then there is a connected subset of U that contains the points x, y (see, e.g. Wilder [27]; Willard [28, p. 201]; Croom [9] or Munkres [20]). It is said to be *locally connected im kleinen* if it has this property for each $x \in X$.

Let $(X, \precsim)$ be a totally ordered set. Suppose that X is order-unbounded, dense, Dedekind-complete and order-separable. Then it is well known by Cantor's famous theorem (see, e.g. Bridges and Mehta [3, Theorem 1.5.8, p. 18]) that X is order-isomorphic with $\mathbb{R}$. Souslin asked if the order-separability condition can be replaced by the countable chain condition (ccc). The *Souslin Hypothesis* (SH) is the assertion that this indeed can be done (see Souslin [25]), i.e. that every order-unbounded continuum which satisfies ccc is order-separable[1].

A total order that is order-unbounded, ccc, dense and Dedekind-complete and which is not order-separable is said to be a *Souslin continuum*. A *Souslin chain* is a totally ordered set which satisfies ccc and has countably many jumps but does not have a real-valued order-preserving representation. It can be proved that the SH holds if and only if there are no Souslin chains (Beardon et al. [1, Proposition 6.4]). It is well-known that the SH is undecidable in ZFC set theory *even* if we assume the Generalized Continuum Hypothesis (GCH). See, e.g. Devlin [10], and [11, p. 109] and Roitman [21, Chapter 7].

To conclude this section, we include the following lemma which is fundamental in the representation theory of topological relational structures.

Lemma 2.1. *Let $(X, \precsim)$ be a nontrivial totally preordered set and f a real-valued order-embedding on X. Then there is a τ-continuous real-valued order-embedding on X in any topology τ that is finer than the order topology on X.*

Proof. See Theorem 3.2.2 in [3]. □

[1] There are several equivalent versions of the SH. For example, the SH may also be stated as the assertion that there are no Souslin trees, i.e. uncountable trees such that all branches and all anti-chains (in particular, levels) are countable. The SH may also be defined as the assertion that there does not exist a Souslin Line, i.e. an order-unbounded ccc continuum which is topologically nonseparable. There are also other versions (see, e.g. Willard [28, p. 158]). The SH is also related to the existence of weakly compact cardinals (see Jensen [17]). For more on the Souslin problem see Rudin [22–24].

3. Locally connected separable topologies are representable

We prove first that the assertion that a locally connected topology which satisfies ccc has the CRP is undecidable in ZFC. To that end, we start by proving the following lemma which is a generalization of Corollary 6.10 in Beardon et al. [1].

Lemma 3.1. *Let $(X, \precsim)$ be a totally ordered set with countably many jumps and suppose that the SH holds. If X satisfies ccc then it follows that there is a continuous order-preserving function on X in any topology that is finer than the order topology.*

Proof. Since the SH holds it follows that there are no Souslin chains (Beardon et al. [1, Proposition 6.4]). If X is nonrepresentable it follows by the Structure Theorem (Beardon et al. [1, Theorem 6.7]) that X is a either a long chain or a planar chain or an Aronszajn-like chain. But each of these chains does not satisfy ccc. Therefore, we conclude that X is representable. We then apply Lemma 2.1 and the result is proved. □

We now prove the following theorem about locally connected topologies that satisfy ccc. In order to understand the significance of the theorem proved below we observe first, as stated above, that the SH is undecidable in ZFC. It is important to observe that there is no presumption or justification for thinking (just as in the case of the Continuum Hypothesis CH) that either the SH or its negation is likely to be true (see, e.g. Todorčević [26], Roitman [21] or Ciesielski [8]). For example, if we assume the axiom of constructibility then there exist Souslin trees and therefore the SH is false (Devlin [11, p. 108–177]); on the other hand, if we assume Martin's axiom and not-CH then the SH is true (see Roitman [21, p. 121]).

Theorem 3.2. *Let (X, τ) be a locally connected space that satisfies ccc. Then the assertion that τ has the* CRP *is undecidable in* ZFC.

Proof. Suppose first that the SH holds. Let $\precsim$ be a τ-continuous total preorder on X which we may assume is a total order because both local connectedness and ccc are divisible properties so that they also hold for quotient spaces. Since X is locally connected each component V is an open set. Now because X satisfies ccc we conclude that the family of components $\mathcal{V} = (V_i)_{i \in \mathbb{N}}$ is countable.

Next, we claim that X has only countably many jumps. Indeed, suppose that (a, b) is a jump in X with $a \prec b$. Then clearly both the points do not belong to V_n for any n because V_n is connected and the preorder is τ-continuous. Therefore, there exist $m \neq n$ such that $a \in V_m$ and $b \in V_n$. This implies that a is the last element of V_n and b the first element of V_n. To prove this suppose that a is not the last element of V_m. Then there exists an element, say $z \in V_n$ such that $a \prec z$. Now we cannot have $a \prec z \prec b$ since (a, b) is a jump. On the other hand, if $b \prec z$ then the element $b \notin V_m$ clearly disconnects V_m since the preorder is τ-continuous. This is a contradiction because V_m is connected. We have proved that a is the last element of V_m. Similarly we see that b is the first element of V_n. This procedure defines an injection from the set of all jumps into the countable set of pairs of elements of a countable set $\mathcal{V}$ and the proof of the claim is finished.

Therefore, all the conditions of Lemma 3.1 are satisfied. We conclude that if the SH holds then every locally connected topological space which satisfies ccc has the CRP.

Suppose now that SH does not hold. Then we claim that there exists a locally connected space X which satisfies the countable chain condition (ccc) but which fails to satisfy the continuous representation property (CRP). Indeed, since the SH does not hold there exists a Souslin chain say, X. That is, X is a nonrepresentable chain that satisfies the countable chain condition (ccc) and has only countably many jumps (a_n, b_n) $(n \in \mathbb{N})$ (see, Beardon et al. [1]). The space X need not be locally connected in the order topology because a point, say, $c \in X$ may be an accumulation point of a sequence of jumps and will, therefore, not have a neighbourhood basis consisting on connected open subsets. Therefore, between each such jump we interpose a copy of the rationals $\mathbb{Q}$ to get a chain which is densely ordered. Let Y be the Dedekind completion of the chain X. This Dedekind completion is essentially unique (see, e.g. Gillman and Jerison [15, p. 3]), and it is clear that a chain is representable by a real-valued order-embedding if and only if its Dedekind completion is representable. We claim that Y is locally connected in its order topology. To prove this let c be an arbitrary element of Y and let U be an open neighbourhood of c. Since Y has no jumps, the point c cannot be an accumulation point of a sequence of jumps, which would prevent Y from being locally connected. Suppose, to the contrary, that the point c does not have a local basis of connected open subsets. Then there exists some open neighbourhood V of c that is not connected. This can happen, by the properties of the order topology, if and only if the open set V has jumps or fails to be Dedekind-complete (see, e.g. Jameson [16, p. 52]) which, by construction, is impossible. Therefore, the supposition is false and Y is locally connected. Finally, Y is not representable because $X \subseteq Y$ is a nonrepresentable chain.

We have proved that if the SH does not hold then there is a locally connected topological space which satisfies ccc but which does not have an order-preserving representation. Since the SH is undecidable in ZFC (see, e.g. Roitman [21]) the proof of the theorem is finished. □

In view of the above theorem it is natural to look for other topological conditions which enable us to get a positive result. One such condition is the topological separability condition. We prove below that a locally connected separable topology has the CRP.

Observe that if X is a locally connected separable topological space, and $\precsim$ is a continuous total preorder on X, then the quotient space through indifference $Y = X/\sim$, endowed with the quotient topology, is locally connected (see e.g. Dugundji [12, p. 125]) and separable (see, Dugundji [12, p. 175]). Moreover, the total order that $\precsim$ induces on Y is obviously continuous. A continuous representation for this total order induces a continuous representation for $\precsim$. Therefore, to prove that a locally connected separable topology is representable it is enough to prove that every continuous total order has a continuous representation.

In the theorem to follow the proof is based on the concept of an *inductive limit* whose definition we include here for the sake of completeness.

Definition. A *directed set* $(D, \precsim)$ is a preordered set with the following property: For each $a, b \in D$ there exists an element $c \in D$ such that $a \precsim c$ and $b \precsim c$.

Let D be a directed set and let $\{X_\alpha\colon \alpha \in D\}$ be a family of topological spaces, indexed by D. For each pair of indices α, β satisfying $\alpha \precsim \beta$ assume that there is given a continuous map $\phi_{\alpha,\beta}: X_\alpha \to X_\beta$, and that these maps satisfy the conditions: (i) $\phi_{\alpha,\alpha}$ is the identity on X_α; (ii) if $\alpha \precsim \beta \precsim \gamma$, then $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma} \circ \phi_{\alpha,\beta}$.

Then this family $\{X_\alpha, \phi_{\alpha,\beta}\}$ of spaces and maps is called an *inductive spectrum* over D, with spaces X_α and connecting maps $\phi_{\alpha,\beta}$.

The image of an element $x_\alpha \in X_\alpha$ under any connecting map is termed a *successor* of x_α. Each inductive spectrum $\{X_\alpha, \phi_{\alpha,\beta}\}$ yields a limit space in the following way: Let $S = \Sigma\{X_\alpha\colon \alpha \in D\}$ be the free union (see, e.g. Dugundji [12, p. 127]) of the spaces endowed with the final topology relative to the inclusions $\iota_\alpha : X_\alpha \to S$. Two elements $x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$ in S are said to be *equivalent* whenever they have a common successor in the spectrum. The quotient space of S through this equivalence is called the *inductive limit* of the spectrum, and denoted by $\varinjlim X_\alpha$. On this space we shall consider the quotient topology.

A particular case appears if $X_\alpha \subseteq X_\beta$ whenever $\alpha \precsim \beta$, and the connecting maps are the inclusions. In this case, the inductive limit is said to be the *inclusion inductive limit* denoted $\lim_{\subseteq} X_\alpha$.

If we have a family of continuous maps $\{f_\alpha : X_\alpha \to Y;\ \alpha \in D\}$ from the elements of an inductive spectrum $\{X_\alpha, \phi_{\alpha,\beta}\}$, with values on a topological space Y, such that they are *compatible* ($f_\alpha = f_\beta \circ \phi_{\alpha,\beta}$), then, by the *universal property of inductive limits* (see Dugundji [12, pp. 421–422]) there exists a continuous map $f : \varinjlim X_\alpha \to Y$ such that $f_{|X_\alpha} = f_\alpha$ ($\alpha \in D$).

Theorem 3.3. *Every weakly locally connected separable topology has the* CRP.

Proof. Let (X, τ) be a topological space with τ weakly locally connected and separable. Let $\precsim$ be a τ-continuous total preorder defined on X. Once again the quotient space through indifference $Y = X/\sim$ is separable (see Dugundji [12, p. 175]), the total order that $\precsim$ induces on Y is obviously continuous, and a continuous representation for this total order induces a continuous representation for $\precsim$. Let $p : X \to Y$ denote the canonical projection onto the quotient. If A is a connected component of Y, and $x \in p^{-1}(A) \subseteq X$, we can consider the component U to which x belongs. By hypothesis, U is open. Moreover, $p(U)$ is connected because p is continuous.

Hence there exists some connected component $B \subseteq Y = X/\sim$ such that $p(U) \subseteq B$. But $p(x) \in A \cap B$ and A was also a connected component of Y. This implies $A = B$ and $p(U) \subseteq A$. Thus $U \subseteq p^{-1}(A)$, and U is an open neighbourhood of x. Therefore $p^{-1}(A)$ is τ-open and equivalently A is open in the quotient topology.

As a consequence, working if necessary on a quotient space, we can assume without loss of generality that $\precsim$ is actually a continuous total order.

Since the space is separable and the connected components are open, it follows that there are at most countably many connected components. Thus we can write X as the disjoint union of its open components, that is, $X = \bigcup (X_n)_{n\in\mathbb{N}}$ where each X_n denotes a connected component of X (in particular $X_n \cap X_m = \emptyset$ $(n \neq m)$). Now given $n \neq m$ let us see that either $x_n \prec x_m$ for every $x_n \in X_n$, $x_m \in X_m$ or vice versa. In other words $\precsim$ naturally induces a total ordering on the family of components of X. Actually, if $x_n \prec x_m \prec y_n$ for some $x_n, y_n \in X_n$ and $x_m \in X_m$, it follows that $X_n = (L(x_m) \cap X_n) \cup (U(x_m) \cap X_n)$, which contradicts the connectedness of X_n since $(L(x_m) \cap X_n)$ and $(U(x_m) \cap X_n)$ are open and nonempty because $x_n \in L(x_m)$ and $y_n \in U(x_m)$. Similarly, the symmetric situation $x_m \prec x_n \prec y_m$ for some $x_n \in X_n$ and x_m, $y_m \in X_m$ also leads to a contradiction.

Given a component X_n of X, such a component is connected by definition, and separable (see Dugundji [12, p. 175]). Also, the restriction of $\precsim$ to each component is obviously a continuous preorder on such component. Thus by Eilenberg's classical theorem [13] we can find a continuous and strictly order-preserving map $f_n : X_n \to \mathbb{R}$. We can proceed inductively in order to construct a continuous order-preserving representation on the whole X. To do so, once a continuous and strictly order-preserving map $f_1 : X_1 \to \mathbb{R}$ has been constructed, mapping X_1 on an interval $(\alpha_1, \beta_1) \subset \mathbb{R}$, it may occur that $x_2 \prec x_1$ for every $x_1 \in X_1$ and $x_2 \in X_2$, or vice versa. In the first situation we can define a continuous and strictly order-preserving embedding $f_2 : X_2 \to \mathbb{R}$ that maps X_2 on an interval $(\alpha_2, \beta_2) \subset \mathbb{R}$ such that β_2 is strictly smaller than α_1. In the second situation we can define f_2 taking values in an interval (α_2, β_2) with $\beta_1 < \alpha_2$. Observe that on the disjoint union $X_1 \cup X_2$ the map $g_2 : X_1 \cup X_2 \to \mathbb{R}$ given by $g_2(x) = f_1(x)$ if $x \in X_1$, and $g_2(x) = f_2(x)$ if $x \in X_2$, is well-defined and continuous because X_1 and X_2 do not meet.

In the same way, if $X_1 \prec X_2$, we may locate the component X_3 with respect to $X_1 \cup X_2$ as $X_1 \prec X_2 \prec X_3$ or $X_3 \prec X_1 \prec X_2$ or else $X_1 \prec X_3 \prec X_2$ and proceed to define a continuous order-embedding g_3 on $X_1 \cup X_2 \cup X_3$.

This process goes on indefinitely as in the Cantor back-and-forth method, so that for each $k \in \mathbb{N}$ we have a continuous and strictly order-preserving map $g_k : \bigcup_{i=1}^{k} X_i \to \mathbb{R}$ such that the restriction of g_k to $\bigcup_{i=1}^{k-1} X_i$ is g_{k-1}. Now the map $g : X \to \mathbb{R}$, given by $g(x) = g_k(x)$ provided that x lies in X_k, is well-defined and strictly order-preserving. Finally, it is also continuous because, the union $X = \bigcup_{i\in\mathbb{N}} X_i$ being disjoint, and each X_i being an open component for any $i \in \mathbb{N}$, it holds that, calling $Y_k = \bigcup_{i=1}^{k} X_i$ $(k \in \mathbb{N})$, the set X has the inductive topology relative to the increasing family of subsets $(Y_k)_{k\in\mathbb{N}}$ (see Candeal et al. [6] for details).

This concludes the proof. □

Corollary 3.4. *Every locally connected separable topology has the* CRP.

Remark 3.5. The converse of Corollary 3.4 is not true: Consider, for instance, the set $X = \{(0,0)\} \cup \{(x, \sin(\frac{1}{x})) \colon x \in (0,1]\} \subset \mathbb{R}^2$, endowed with the restriction of the Euclidean (metric) topology of the plane. Since X is metric and separable, the metric topology is representable (see Candeal et al. [5]). On the other hand, it is plain that X is weakly locally connected because it is indeed connected. However, it is not locally connected.

Remark 3.6. We see that Theorem 3.3 generalizes the classical Eilenberg's theorem [13]. The result in Theorem 3.3 can also be generalized in the following way: We can prove that if a separable topological space (X, τ) has a countable number of connected components, then (X, τ) satisfies the continuous representability property (CRP). We prove this claim in the next Proposition 3.7.

Proposition 3.7. *Let (X, τ) be a separable topological space such that X can be partitioned as a countable union $X = \bigcup_{n\in\mathbb{N}} X_n$ of pairwise disjoint connected components. Then every τ-continuous total preorder defined on X is representable through a continuous order-embedding.*

Proof. Let $\precsim$ be a τ-continuous total preorder defined on X. Since X is τ-separable, there exist a countable subset $D \subseteq X$ such that D meets every τ-open subset of X. Define now a jump in $(X, \precsim)$ as a couple (x, y) where $x, y \in X$, $x \prec y$ and the subset $\{z \in X \colon x \prec z \prec y\}$ is empty. Two jumps (x, y) and (a, b) are said to be equivalent if $x \sim a$ and

$y \sim b$. Now observe that the family of non-equivalent jumps of $(X, \precsim)$ must be countable because $\precsim$ is continuous, the number of connected components of X is countable, and each jump (x, y) provokes a decomposition of X in two nonempty clopen subsets, namely $L(y)$ and $U(x)$. Let $\{(x_k, y_k)\colon k \in \mathbb{N}\}$ the (countable) family of non-equivalent jumps in $(X, \precsim)$. We enlarge the subset D with the elements $\{x_k\colon k \in \mathbb{N}\}$ and $\{y_k\colon k \in \mathbb{N}\}$ that correspond to the aforementioned family of jumps. being $E = D \cup \{x_k\colon k \in \mathbb{N}\} \cup \{y_k\colon k \in \mathbb{N}\}$, it is plain that E is a countable subset of X. Finally, it is straightforward to see that E is order-dense in $(X, \precsim)$ so that $(X, \precsim)$ is order-separable, hence representable, and as a consequence of Lemma 2.1, the structure $(X, \precsim)$ is indeed continuously representable. $\square$

Question 3.8. Are there other topological conditions (perhaps more general than the separability condition used in Corollary 3.4 above) that we can use to obtain continuous representability?

One such condition is the following: A topological space (X, τ) has caliber κ, where κ is a cardinal if, and only if, whenever $\mathcal{U}$ is a family of open subsets of X with cardinality κ there is a subfamily $\mathcal{V}$ of the same cardinality such that the intersection of all members of the subfamily is nonempty. Hence, the question arises if separability in Corollary 3.4 can be replaced by the condition that X has caliber $\aleph_1$.

This question is important for two reasons. First, the condition for a topological space to have caliber $\aleph_1$ is intermediate between the countable chain condition and the topological separability condition that have been used above (see Engelking [14, p. 157]). Therefore, it is particularly interesting to study the properties of a topological space X that is locally connected and has caliber $\aleph_1$. Second, the caliber of a topological space X is closely related to an important cardinal function in topology called the Šanin number $s(X)$ of X. In fact, $s(X)$ is the smallest cardinal κ such that its successor cardinal is the caliber of X.

One other possible condition is "local separability": A topological space (X, τ) is said to be *locally separable* if every point $x \in X$ has a τ-open neighborhood that is separable in its relative topology. Observe that a locally separable and locally connected topological space has the property of "local continuous representability" in the sense that given a continuous total preorder and a point, there exists an open neighbourhood of this point where the restriction of the preorder is continuously representable, so that we have (continuous) local representations, but perhaps not a global representation defined on the whole space. In general, locally separable plus locally connected topologies do not satisfy CRP (an example is $(0, 1) \times (0, 1)$ endowed with the topology induced by the lexicographic order). However, in some special cases, it is interesting to try to look for theorems about "gluing" local representations of preorders to get a global representation as is done in paragraph 4.1 of Candeal et al. [6].

4. Other representability properties for locally connected topologies

A topological space (X, τ) is said to have the *hereditary continuous representation property* (HCRP) if every subset $Y \subseteq X$ has the continuous representation property (CRP) with respect to the topology τ_Y that τ induces on Y.

It is obvious that HCRP *implies* CRP. But *the converse is not true*, as next example shows.

Example 4.1. Let X be an uncountable set. Select a point $p \in X$. Consider on X the topology τ_p of "point p included", that is, given $A \subseteq X$, $A \in \tau_p \iff A = \emptyset$ or else $p \in A$. It is straightforward to see that the only continuous total preorder on X is the trivial one that declares indifferent all the elements belonging to X. Therefore (X, τ_p) trivially satisfies CRP.

However, $X \setminus \{p\}$ inherits the discrete topology, so that any total preorder on $X \setminus \{p\}$ is continuous. If we consider a well-ordering on $X \setminus \{p\}$, it is clear that such well-ordering fails to have a representation (even discontinuous!) in the real line. (See, e.g. Beardon et al. [1] for a further account.) Consequently, $(X \setminus \{p\}$, discrete topology) does not satisfy CRP, so that (X, τ_p) does not satisfy HCRP.

Remark 4.2. Since the last Example 4.1 of an uncountable set X endowed with the "point included" topology corresponds to a *locally connected plus separable* topological space, it follows that: *It is not true, in general, that locally connected plus separable topologies satisfy the hereditary continuous representation property* (HCRP).

Given a topological space (X, τ), the topology τ is said to satisfy the semicontinuous representability property (SRP) if every τ-upper (respectively τ-lower) semicontinuous total preorder $\precsim$ defined on X admits a numerical representation by means of a τ-upper (respectively τ-lower) semicontinuous real-valued order-preserving embedding.

In general *SRP implies CRP, but the converse is not true*, as proved in Proposition 4.4 in Bosi and Herden [2]. This suggests that the main result obtained in Corollary 3.4 *cannot be generalized to the semicontinuous case*, so that such result (Corollary 3.4) is, in this sense, the best one that we can expect. The following two examples prove this assertion.

Example 4.3. As in Example 4.1, let X be an uncountable set, select an element $p \in X$, and consider on X the topology τ_p of "point p included", a topology that is locally connected and separable, so that (X, τ_p) satisfies CRP. However (X, τ_p) does not satisfy SRP: Consider a well-ordering on X whose first element is p. It is obvious that for every $q \in X$, $q \neq p$, every lower contour set $L(q)$ is τ_p-open since it contains p. In addition $L(p) = \emptyset$. Thus, any such well-ordering is τ_p-upper semicontinuous. But it is not representable, not even using a discontinuous order-embedding, so that (X, τ_p) does not satisfy the semicontinuous representation property.

Example 4.4. Being $(X, \|\cdot\|)$ a Banach space endowed with its norm topology, let X^* be the dual Banach space of X. Consider on X^* the weak-star topology $\sigma(X^*, X)$ that we shall denote ω^*. Theorem 4.2 in Campión et al. [4] states that (X^*, ω^*) satisfies SRP if and only if $(X, \|\cdot\|)$ is separable. In addition, a result in Monteiro [19] (see also Theorem 3.2 in Campión et al. [4]) proves that (X^*, ω^*) always satisfies CRP. Consider now the dual $(\ell_\infty)^*$ of the Banach space ℓ_∞. Endow that dual $(\ell_\infty)^*$ with the weak-star (ω^*) topology. Then, it follows that $((\ell_\infty)^*, \omega^*)$ is a locally connected plus separable topological space that satisfies CRP but not SRP.

5. Generalizations of locally connected topologies

Generalizations of locally connected spaces are very useful in topology (see, e.g. Knaster and Kuratowski [18]), topology of manifolds (see, e.g. Wilder [27]) and in other areas. This motivates the importance of the material to follow.

We start by recalling that a topological space (X, t) is said to be *locally connected im kleinen at the point x* if for each open neighbourhood U of x there is a neighbourhood V of x such that if $y \in V$ then there is a connected subset of U that contains x, y (see, e.g. Croom [9] and Wilder [27]). It is said to be locally connected im kleinen if it has this property for each $x \in X$. It is known that if a space is locally connected at x then it is locally connected im kleinen at x but the converse is false (consider e.g. an infinite sequence of broom spaces, as in Croom [9]). On the other hand, it is known that if a space is locally connected im kleinen at all points then it is locally connected.

We now prove the following theorem.

Theorem 5.1. *Let (X, t) be a separable topological space equipped with a continuous total preorder $\precsim$. Consider the set of all points $s \in X$ with the property that s has either an immediate predecessor or an immediate successor or both. Let S be the subset of all such points and suppose that X is locally connected im kleinen at each point of S. Then there exists a continuous order-preserving function on (X, t).*

Proof. Since X is a separable topological space there is a countable dense subset Z of X which we now consider as fixed. Consider the set J of all pairs (x, y) of points of X such that $x \prec y$ and there is no point $z \in Z$ such that $x \prec z \prec y$. Then the set S is merely the union of all such pairs of points.

We claim that the set S is isolated. If x is a nonisolated point of S then there is a net (x_α) of points of S such that the net converges to x. Since $x \in S$ and, by hypothesis, the space X is locally connected im kleinen at x for every open neighbourhood U of x there is a neighbourhood V of x such that if for any point $y \in V$ there is a connected subset of U that contains both the points x, y. Since the net (x_α) converges to x it is eventually in every such neighbourhood V of x. But any two distinct points of V cannot belong to a connected subset of U by definition of the set S, i.e. the pair (x, y) would disconnect U. But this contradicts the hypothesis that X is locally connected im kleinen at x. Therefore, we conclude that S is isolated. Since S is isolated it cannot contain a copy of the rationals $\mathbb{Q}$ so that it is a scattered set.

We prove next that S is a short total preorder, that is, there is no order-embedding of the first uncountable ordinal or its dual into S. Indeed, if S is long then by Theorem 3.1 in Beardon et al. [1] there exists an uncountable well-ordered strictly decreasing family of order-intervals of S. But this would imply that there is an uncountable family of nonempty pairwise disjoint open intervals which is impossible because X is a separable topological space and the preorder is continuous. Therefore, we may conclude that S is a short total preorder.

We claim that S is countable. Indeed, if S were uncountable, then by Lemma 6.6 in Beardon et al. [1] S contains an uncountable dense sub-chain, say, T. In particular, since T is dense, it contains a copy of the rationals $\mathbb{Q}$ so that, in particular, T is not scattered. But this contradicts the conclusion that S is an isolated and scattered set. Hence, we have proved that S is countable.

Since X is a separable topological space there is a countable dense subset Z. Now let $W = Z \cup S$. We have proved above that S is countable and so it follows that W is countable. Now it is easily verified that the set W is order-dense in X. It follows from Theorem 3.2.9 on p. 47 of Bridges and Mehta [3] that there is a continuous order-preserving function on X and the theorem is proved. □

Remark 5.2.

(a) In the above Theorem 5.1 we do not assume that X is locally connected so that it may not be true that the component of each open set in X is open. This will happen, e.g. if X is the ordered sum of two spaces A, B such that B is as in the theorem above and A is a continuously representable space that is not locally connected (see, e.g. Candeal et al. [7]). In particular, the components of X need not be open sets. Therefore, the above theorem is a generalization of Theorem 1 in Candeal et al. [7]. In particular, it is an alternative proof of Theorem 1 in that paper.

(b) It is important to observe that, in Theorem 5.1, X is only assumed to be locally connected im kleinen for points in a *proper* subset of X, namely, at points of S. These points are also not dense in X as is easily verified.

Acknowledgement

This paper was presented at the International Mediterranean Congress of Mathematics (CIMMA 2005) held in Almería, Spain in June 2005. Also, the ideas introduced in this paper were discussed with other participants in the Sixth Iberoamerican Congress of Topology and its Applications (CITA 2005), held in Puebla, Mexico in July 2005. The authors are indebted to the colleagues that took part in the sections of Topology during such meetings. This work has been partially supported by the Research Grant MTM2006-15025 (Spain).

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